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Inflationary reheating classes via spectral methods

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Inflationary reheating is almost completely controlled by the Floquet indices, μ_k. Using spectral theory, we demonstrate that the stability bands (where $\mu_k = 0$) of the Mathieu and Lamé equations are destroyed even in Minkowski spacetime, leaving a fractal Cantor set or a measure zero set of stable modes in the cases, where the inflaton evolves in an almost-periodic or stochastic manner, respectively. These two types of potential model the expected multi-field and quantum back reaction effects during reheating. [S0556-2821(98)50414-6]

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I. INTRODUCTION

Inflation is perhaps the heir apparent in high energy cosmology. No other theory can be cast successfully in so many guises. Yet, to be successful, any inflationary scenario must reheat the universe. This epoch is typically very short and has become a powerful virtual laboratory for researching non-equilibrium quantum field theory in curved spacetime [1–11]. It is defined by rapidly evolving paradigms (see the left half of Fig. 1), attempting to deal with the non-perturbative processes occurring in preheating [1]. The defining characteristic of a preheating model are its Floquet indices μ_k, which control the number of produced particles $n_k \sim \int dk k^2 e^{2m\mu_k t}$ and variances of the fields [9].

Of particular interest are the issues of how the μ_k vary as (i) the parameters (typically the coupling constants) of the theory are changed, and (ii) the functional form of the inflaton evolution is altered. The first has been studied in depth in the case, where the inflaton evolution is exactly periodic and is the very origin of preheating. The second issue is much less studied and is the subject of this paper, where we introduce spectral methods as an elegant classification tool for inflationary reheating classes; though it also applies to the parametric amplification of gravitational waves [11]. To illustrate the idea, consider the Hill equation

$$\ddot{y} + [A - 2qP(2t)]y = 0 \tag{1}$$

where $P(2t)$ is any periodic function of the independent variable t. When $P(\cdot) \propto \cos(\cdot)$, we have the Mathieu equation, while $P(\cdot) = \mathrm{cn}^2(\cdot)$ yields the Lamé equation, where $\mathrm{cn}(\cdot)$ is the elliptic cosine function. Now Eq. (1) is equivalent to the one-dimensional Schrödinger operator-eigenvalue problem:

$$\mathcal{L}(y) \equiv -\frac{d^2 y}{dx^2} + Q(x)y = \lambda y \tag{2}$$

under the transformations $x \leftrightarrow t$, $Q(x) \leftrightarrow 2qP(2t)$, $\lambda \leftrightarrow A$. We denote the spectrum of $\mathcal{L}$ by $\sigma(\mathcal{L})$, its complement by $\overline{\sigma(\mathcal{L})}$. The crucial point to notice is that the set of modes with positive Floquet index of Eq. (1) exactly coincides with $\overline{\sigma(\mathcal{L})}$ of Eq. (2). By exploiting this equivalence, we will show that:

(I) Generically for almost-periodic potentials $Q(x)$, the corresponding spectrum is a fractal Cantor set. This implies a Floquet index which is positive on a dense subset of momenta $k \in [0,\infty)$. As the strength of the coupling is varied, drastic changes to the spectrum can occur that are absent in the purely periodic case.

(II) Preheating exists in stochastic inflation and the Floquet index is positive with Lebesgue measure 1 for all momenta k. Estimates for μ_k are given in the perturbative and non-perturbative limits by Eqs. (15) and (16), respectively.

These results describe geometrically the breakup of the stability-instability chart that in the case of the Hill equation (1) has a neat band-structure. Visually, this is similar to the breakup of invariant tori in the Kolmogorov-Arnold-Moser (KAM) theory of chaos [12]. Similar results to those of II above have recently been found independently [13,14] in specific cases for the noise. In particular, Zanchin *et al.* [14] rewrite Eq. (1) as a first order matrix equation and then use Furstenberg's theorem [15] regarding products of independent, identically distributed random matrices to obtain estimates of μ_k, showing that noise increases the μ_k over the periodic case and that $\mu_k > 0$ for all wavelengths, thus also demonstrating the break-up of the stability bands, although in the context of small noise.

The results of I and II above enlarge the known reheating territory (see Fig. 1). The classical theory of reheating was developed in the early 1980s [16], with recognition of the importance of perturbative resonances [17]. In this case the resonance bands essentially dwindle into a discrete sequence

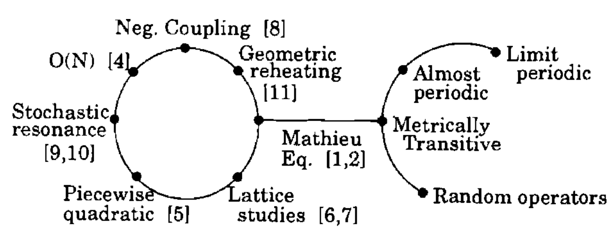

FIG. 1. A schematic map of models and approximations in preheating. Minimal references for techniques in the exactly periodic case are shown in brackets. The right-hand branch corresponds to paradigms developed in this paper.

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of lines. The paradigm shift of preheating [1], showed that the resonance bands could be very broad at large q, a fact that extends to gravitational waves [11,18]. This was followed by approximations to deal with the non-perturbative effects [5,4], and the idea of non-thermal symmetry restoration induced by the large quantum fluctuations [3]. Recently stochastic resonance [9,10] has represented a shift away from the simple picture of static resonance bands, due to the large phase fluctuations which, mod 2π, behave randomly in the non-perturbative regime in an expanding universe. Finally, the negative-coupling instability [8] represents the exciting possibility that modes can evolve where the spectrum of the corresponding Schrödinger operator is almost empty,[1] and Floquet indices are very large. This mechanism has been realized within the framework of non-minimal fields coupled only to gravity, which therefore offers a geometric reheating channel [11] due to the oscillation of the Ricci curvature.

In this paper we consider almost-periodic (Sec. II) and random (Sec. III) potentials which are both realizations of metrically transitive (i.e., ergodic and homogeneous) operators [20,21].

II. CANTOR REHEATING

We limit our discussion in this section to Minkowski spacetime.[2] Consider the inflaton, ϕ, and the two minimally coupled scalar fields φ, χ with the natural mass hierarchy $m_\phi \gg m_\varphi \gg m_\chi$. In this section we consider the effective potential:

$$V(\phi,\varphi,\chi)=\frac{m_\phi^2}{2}\phi^2+\frac{\lambda}{4}\phi^4+\frac{m_\varphi^2}{2}\varphi^2+\frac{m_\chi^2}{2}\chi^2+\frac{g^2}{2}F(\phi)\chi^2+\frac{h^2}{2}J(\varphi)\chi^2 \tag{3}$$

$F(\cdot),J(\cdot)$ are assumed to be analytic in their respective arguments. The evolution of the χ_k modes is then [1]

$$\ddot{\chi}_k+\left(\frac{k^2}{a^2}+m_\chi^2+g^2F(\phi)+h^2J(\varphi)\right)\chi_k=0. \tag{4}$$

Instead the inflaton zero-mode evolves according to

$$\ddot{\phi}+m_{\phi,eff}^2\phi+\lambda\phi^3=0 \tag{5}$$

where the frequency of oscillation is partially controlled by the effective mass:

$$m_{\phi,eff}^2=m_\phi^2+g^2\frac{F'}{\phi}\langle\chi^2\rangle+3\lambda\langle\delta\phi^2\rangle. \tag{6}$$

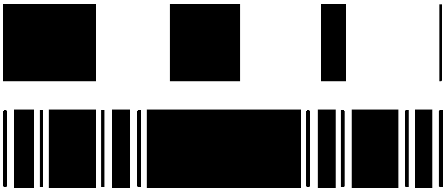

FIG. 2. A schematic diagram comparing the spectrum of the Mathieu equation (upper) and that of a generic limit-periodic operator with fractal Cantor spectrum (lower). The black shading represents the complement of the spectrum, i.e., modes with $\mu_k \neq 0$. The Cantor spectrum is only shown at second level due to resolution limitations. The same pattern repeats itself in each white gap and hence the number of stable modes is much smaller than in the periodic case.

In the case that $\langle\chi^2\rangle=\langle\delta\phi^2\rangle=0$, the inflaton simply oscillates with constant period. However, when there are multiple-fields, or the mass acquires corrections due to quantum fluctuations, the period is no-longer constant, but may increase monotonically, oscillate or exhibit random fluctuations. As an example, in the special case of a Yukawa interaction, $F(\phi)=\phi$, Eq. (5) has a pure driving term $\propto g^2\langle\chi^2\rangle$.

Understanding the behavior of the Floquet indices in these more general cases is extremely complex and we, therefore, search for special, solvable classes, which naturally leads to the study of almost-periodic potentials [22]. In this case, the spectral theory becomes significantly richer and more beautiful.

In general, for a separable Hilbert space, as we have here, the spectrum can be decomposed with respect to an abstract measure $d\mu$ as $\sigma=\sigma_{AC}\cup\sigma_P\cup\sigma_{SC}$ [20], where σ_{AC} is the absolutely continuous, σ_P is the pure point and σ_{SC} the singular continuous part of the spectrum. An important constraint is that the Floquet index must vanish for $\lambda\in\sigma_{AC}$, i.e., $\sigma_{AC}=\{\lambda\in\mathbf{R}|\mu_\lambda=0\}$ bar a set of measure zero [20], and hence σ_{AC} corresponds to modes with bounded evolution (extended eigenfunctions). On the complement, $\overline{\sigma_{AC}}$, the Floquet indices are known to be positive, and hence in the case when σ_{AC} is empty, $\mu_k>0$ almost everywhere [23].

To understand this better, consider the periodic potential. In this case, $\sigma(\mathcal{L})=\sigma_{AC}$ and on $\overline{\sigma(\mathcal{L})}$, $\mu_\lambda>0$. It is the fact that σ_{AC} is so large that forces the strong band structure of the Mathieu and Lamé equations. In the Lamé case, $\overline{\sigma(\mathcal{L})}$ consists of a *single* band and there are very few exponentially growing modes [4]. A comparison between the archetypal Cantor set and the spectrum of the Mathieu equation is shown in Fig. (2).

The almost-periodic potentials are those for which the Fourier transform consists of a frequency basis $\{\omega_i\}$ where the smallest vector space containing this basis, $\mathcal{M}$, is dense in $\mathbf{R}$. If $\mathcal{M}$ is generated by finitely many ω_i, then the potential, $Q(t)$, is quasi-periodic, i.e., $Q(t)=f(\omega_1 t,\ldots,\omega_n t)$ with $f(t_1+m_1,\ldots,t_n+m_n)=f(t_1,\ldots,t_n)$ with $m_i\in\mathbf{Z}$ and the ω_i pairwise incommensurate [22].

As a simple example, consider Eq. (4) with $F(\phi)=\phi^2$, $J(\varphi)=\varphi^2$ and $\lambda=0$. Then the equations for the quantum fluctuations of χ_k are

[1] This can be restated as the fact that the integrated density of states, proportional to the rotation number [19], vanishes.

[2] The general expanding Friedmann-Lemaitre-Robertson-Walker (FLRW) case is contained within the spectral theory of Sturm-Liouville operators.

$$\ddot{\chi}_k+\left(\frac{k^2}{a^2}+m_\chi^2+g^2\phi^2+h^2\varphi^2\right)\chi_k=0 \quad (7)$$

with $\phi\sim\sin(m_\phi t)$ and $\varphi\sim\sin(m_\varphi t)$. When m_ϕ/m_φ is irrational, the potential is quasi-periodic. The spectrum of Eq. (2) for almost-periodic potentials can be pure point and is, in a non-rigorous way, generally a nowhere dense Cantor set [24]. Hence, in the example above, for infinitely many irrational values of m_ϕ/m_φ, we may expect that the spectrum of χ_k will be nowhere dense, and consequently that almost all modes will grow exponentially.

However, unlike the random potentials to be considered in Sec. (III), the Lebesgue measure of the spectrum, even if a Cantor set, need not be zero [22] and σ_{AC} may be non-empty. Indeed it is possible to have $\sigma=\sigma_{AC}$ [25].

More rigorous results exist in the case that Q is limit-periodic, i.e., it is a uniform limit of periodic potentials. A typical example is provided by

$$Q(t)=\sum_{n=1}^{\infty} a_n\cos\left(\frac{2\pi t}{2^n}\right), \quad \sum |a_n|<\infty. \quad (8)$$

For these potentials we can call on rigorous theorems:

Theorem 1 [25]. $\sigma(\mathcal{L})$ of Eq. (2) is generically[3] *a nowhere dense Cantor set for Q an element of the space of limit periodic potentials. Hence $\overline{\sigma(\mathcal{L})}$ is dense in* $\mathbf{R}$.

Thus the set of k for which $\mu_k\neq 0$, is dense in $\mathbf{R}^+$ despite the fact that σ_{AC} is not empty. This existence of only a Cantor set of stable modes leads us to call this cantor reheating. An associated issue is what happens as the coupling to the potential [analogous to q in Eq. (1)] is increased. While in Eq. (1) this simply changes the breadth of the instability bands and magnitude of μ_k while always leaving $\sigma=\sigma_{AC}$, in the almost-periodic case this is not the case. Indeed, as in the example below, the nature of the spectrum (with respect to the splitting into $\sigma_P,\sigma_{SC},\sigma_{AC}$) can change suddenly with q.

A. The discrete almost-Mathieu equation

Consider the discretized Schrödinger Eq. (2), which is also used (in higher dimensions) in numerical preheating studies [6,7]. A special case is the almost-Mathieu equation with exhibits a rich variety of effects:[4]

$$-y(x+1)-y(x-1)+2q\cos(\alpha x+\omega)y(x)=\lambda y(x). \quad (9)$$

When α is rational, the potential is periodic and one has the band spectrum of the Mathieu equation. However, for irrational α, the spectrum is a Cantor set generically for pairs $(q,\alpha)\in\mathbf{R}^2$ [26]. This shows that the nowhere dense nature of the spectrum is not lost as one increases q and hence moves from perturbative reheating to broad-resonance preheating. Further, when α is irrational, the Floquet index has the lower bound [27]:

$$\mu_k(q)\geqslant\ln|q| \quad (10)$$

so that for $|q|>1$, $\mu_k>0$ and hence the absolutely continuous part of the spectrum, σ_{AC} becomes empty. In this case, if α is a Liouville number,[5] then the spectrum is purely singular continuous, $\sigma=\sigma_{SC}$ [28]. Conversely if $|q|<1$, the point part of the spectrum is absent for irrational α [29].

III. STOCHASTIC INFLATIONARY REHEATING

We now consider the case, where the potential Q in Eq. (2) is random. This models for example, the classical limit of stochastic inflation, where the dynamics of the local order parameter in a FLRW background, are described by [30]

$$\ddot{\phi}+3H\dot{\phi}+V'(\phi)=\frac{H^2}{8\pi^3}V'''(\phi)\xi(t), \quad (11)$$

where $H\equiv\dot{a}/a$ is the stochastically evolving Hubble constant [31], ξ is a colored Gaussian noise of unit amplitude with a correlation time of order H^{-1}, so that the inflaton evolves stochastically. The origin of the noise is the back reaction of quantum fluctuations with wavelengths shorter than the coarse-graining scale [32,31,13].

Here we will consider the potential $V(\phi)=\lambda\phi^4/4$ and as before an interaction term $g^2\phi^2\chi^2/2$. The quantum fluctuations of the fields ϕ and χ are then given by:[6]

$$(a^{3/2}\delta\phi_k)^{\ddot{}}+\left(\frac{k^2}{a^2}+3\lambda\phi^2+\frac{3}{4}p-\frac{3\lambda H^2}{4\pi^3}\xi(t)\right)(a^{3/2}\delta\phi_k)=0, \quad (12)$$

$$(a^{3/2}\chi_k)^{\ddot{}}+\left(\frac{k^2}{a^2}+m_\chi^2+\frac{3}{4}p+g^2\phi^2\right)(a^{3/2}\chi_k)=0, \quad (13)$$

where $p=\kappa[\frac{1}{2}\dot{\phi}^2-V(\phi)]$ is the pressure and $\kappa=8\pi G$. These equations are again equivalent to Eq. (2), but this time with stochastic potentials, which are the opposite extreme to periodic potentials since they generically have empty σ_{AC}. In fact we have the following theorem:

Theorem 2 [33,34]. If $Q(x)$ is a sufficiently random potential for Eq. (2), then $\mu_\lambda>0$ for almost all $\lambda\in\mathbf{R}$ and σ_{AC} of $\mathcal{L}$ is empty with probability one.

[3]The space of limit periodic potentials is a complete metric space and hence *generic* means here "for a dense G_δ," e.g. [24].

[4]For example, Eq. (9) also exhibits the beautiful property of duality, similar to the S-duality of string theory, since under Fourier transform $q\to 1/q$ and $\lambda\to-\lambda/q$ [27].

[5]A Liouville number, α, is irrational, but well approximated by rationals so that there exist integers $p_n,q_n\to\infty$ and a number C with $|\alpha-p_n/q_n|\leqslant Cn^{-q_n}$.

[6]Here we neglect the back reaction of the produced field χ on the expansion; valid during the first phase of reheating before back reaction terminates the resonance.

Indeed we see that a positive Floquet exponent is guaranteed for all modes bar a set of measure zero, and again reheating is significantly different from the periodic case. Here ''sufficiently random'' means nondeterministic [20] and is typically a requirement that correlations decay sufficiently rapidly. An example is a Gaussian random field whose correlation function has compact support. To proceed to obtain quantitative estimates of the Floquet indices, we exploit the fact that Eqs. (12),(13) have the form of stochastic harmonic oscillators.

A. Explicit estimates for the Floquet indices

Consider the stochastic harmonic oscillator, with frequency given by $\omega^2=\kappa^2+q\xi(t)$, where q is a dimensionless coupling to the mean-zero colored stochastic process $\xi(t)$ and $\kappa^2=k^2/a^2$ in our case. The Floquet index has been shown to be strictly positive for all q [35]. It has also been related to the spectral density of fluctuations via averaging over the second moments [36] of the random process ξ [37]:

$$\mu_k=\frac{\int_0^\infty \langle \xi(t)\xi(t-t')\rangle \cos[2\langle \omega^2\rangle t']dt'}{2|\langle \omega^2\rangle|}. \tag{14}$$

This has the form of a fluctuation-dissipation theorem since fluctuations in the inflaton field determine the dissipation rate into other fields. In the case that $\xi(t)$ is a mean zero ergodic Markov process, μ_k can be explicitly estimated as [38]:

$$\mu_k=\frac{\pi}{4}\frac{q^2}{\kappa^2}\hat{f}(2\kappa)+O(q^3) \tag{15}$$

where $\hat{f}$ is the Fourier transform of the expectation value of the two-time correlation function $\langle \xi(t)\xi(t-t')\rangle$. Note that $\mu_k \propto k^{-2}$, so that although all modes grow exponentially, the Floquet index is, as in the periodic and stochastic resonance [9] cases, a rapidly decreasing function of k. In the broad-resonance limit, $q\rightarrow\infty$, we write $\kappa^2=\kappa_0^2+q\kappa_1^2$, which gives (assuming $max\,\xi<\kappa_1^2$) [38]:

$$\mu_k=\frac{\kappa_1}{4\pi}\int_0^{2\pi}d\theta\int d\xi\frac{\sqrt{\kappa_1^2-\xi}}{\kappa_1^2-\xi\cos^2\theta}G[\ln(\kappa_1^2-\xi\cos^2\theta)] + O(1/\sqrt{q}) \tag{16}$$

where G is the infinitesimal generator of $\xi(t)$ defined by the limit of the operator sequence: $G=lim_{t\rightarrow 0}(U_t-\mathbf{I})/t$. Here $\mathbf{I}$ is the identity operator and U_t is a family of operators on the space of bounded continuous functions f, defined by $U_t f(x)=E[f(\xi(t))|\xi(0)=x]$, where E denotes the expectation operator [21]. Again $\mu_k>0$ for all k, and given a model of ϕ evolution, one can explicitly estimate μ_k, and hence the numbers of produced particles n_k and variances $\langle \delta\phi^2\rangle$ and $\langle \chi^2\rangle$.

IV. AMPLIFICATION OF GRAVITATIONAL WAVES

Here it is demonstrated that the evolution equations for gravitons can be cast in the form of the Schrödinger Eq. (2). Within the Bardeen formalism [11], or by using the Weyl tensor, there exists a strong correspondence between gravitational waves and scalar fields during reheating [18]. In the Bardeen approach, gravitational waves are described by transverse-traceless metric perturbations, with mode functions h_k satisfying [11]:

$$(a^{3/2}h_k)\ddot{} + \left(\frac{k^2}{a^2}+\frac{3}{4}p\right)(a^{3/2}h_k)=0, \tag{17}$$

so that $Q\leftrightarrow -3p/4$, $k^2/a^2\leftrightarrow\lambda$ and $a^{3/2}h_k\leftrightarrow y$ establishes the equivalence with Eq. (2). This implies that during reheating with an almost-periodic or stochastic inflaton evolution, all wavelengths will be amplified, rather than just those in the Floquet instability bands. In the stochastic case with small noise, Eq. (15) shows that the tensor spectrum will acquire a tilting with $\mu_k\sim k^{-2}$, causing further subtleties for the inflationary potential reconstruction program.

V. CONCLUDING REMARKS

Using spectral theory, we have shown the breakup of the stability bands of the Mathieu equation. For constant q, the set of stable modes becomes a Cantor set generically for almost-periodic inflaton evolution and a set of measure zero if the evolution is sufficiently random. This implies that quantum corrections to the effective mass will lead to an exponential growth of all modes of the reheated field, and of the gravitational wave background produced during inflation. Further, there exists the possibility, absent in the periodic case, of drastic qualitative changes to the nature of the spectrum at certain transitional values of q, as demonstrated by the discrete almost-Mathieu equation.

This is of relevance to the issue of non-thermal symmetry restoration, since the quantity $\int_{\overline{\sigma(\mathcal{L})\mu_k}}dk$ is much larger in the classes considered here. This should reflect directly in the number of particles produced, n_k, the non-thermality of the spectrum and the variances of the fields.

We have largely limited ourselves here to the study of spectral changes as the potential is changed and found drastic changes from the simple band structure of Floquet theory. However, as a warning, beautiful counter-examples exist, though they are suitably rare. Firstly, using inverse scattering theory for Eq. (2), one can show that there exist, and in fact derive all examples of, almost-periodic potentials which have band spectra similar to the periodic case. An example is given by the Bargmann potentials [19]. Hence it is not always true that the spectrum of an almost-periodic operator must be a Cantor set.

Neither is it true that altering the coupling to the potential always induces a change in the spectrum. From the theory of isospectral deformations of the Schrödinger equation it is known that if $Q(x,y)$ is a potential of Eq. (2) and satisfies the Korteweg–de Vries (KdV) equation $Q_y - 6QQ_x + Q_{xx} = 0$, then $\sigma(\mathcal{L})$ is independent of y and there thus exist 1-parameter families of potentials with the same spectra [19]. The connections with integrable systems become clear, and hence so do the rarity of such events.

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